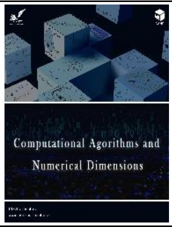
Computational Algorithms and Numerical Dimensions

www.journal-cand.com

Com. Alg. Num. Dim Vol. 1, No. 4 (2022) 130–136.



Paper Type: Original Article

The Modification of the Generalized Gauss-Seidel Iteration Techniques for Absolute Value Equations

Rashid Ali[1,*], Ashfaq Ahmad[1], Ijaz Ahmad[1], Asad Ali[1]

[1] School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan P.R. China; rashidali0887@gmail.com; yousafzai536@yahoo.com; 3420785551@qq.com; asad.adina@gmail.com.

Citation:

Ali, Sh., Ahmad, A., Ahmad, I., & Ali, A. (2022). The modification of the generalized gauss-seidel iteration techniques for absolute value equations. *Computational algorithms and numerical dimensions, 1*(4), 130-136.

Received: 26/08/2022 Reviewed: 26/09/2022 Revised: 02/10/2022 Accept: 19/10/2022

Abstract

This paper proposes two modified generalized Gauss-Seidel iteration techniques to determine the Absolute Value Equations (AVEs). Convergence of the new techniques is established under some appropriate conditions lastly; several numerical examples verify the significance of the techniques.

Keywords: Gauss-seidel iteration techniques, Convergence, Absolute value equations, Numerical examples.

1 | Introduction

In this article, we explore the AVE of the form;

$$Ax - |x| = b. \tag{1}$$

Here $A \in R^{n\times n}, b, x \in R^n$ and $|x|$ represents the vector with the absolute value of each element of x.

The *AVE (1)* has attracted significant interest in the optimization domain for more than twenty years. The most important reason is that they are closely related to Linear Complementarity Problems (LCP) [1], [2] and horizontal LCP [3], which include a vast range of mathematical programming problems and have a wide range of applications [4]-[10].

In recent years, many efforts have been made to develop techniques for finding numerical solutions to the *AVE (1)*. For example, Salkuyeh [11] proposed the Picard-HSS approach to compute *Eq. (1)*. The SOR-like system is discussed in [12]. Additionally, Chen et al. [13] explored various convergence effects for a SOR-like strategy using different parameters. In [15], the authors presented the shift splitting technique to obtain *Eq. (1)* and established the convergence properties of the offered approach. In the study of Moosaei et al. [16], two approaches are offered to determine *Eq (1)*, the Homotopy perturbation approach and the Newton approach with the Armijo step.

Corresponding Author: rashidali0887@gmail.com

https://doi.org/10.22105/cand.2022.157605

Fakharzadeh and Shams [17] offered a Mixed-Type Splitting (MTS) strategy for the calculation of *Eq. (1)* and demonstrated its convergence conditions. In addition, there are several other numerical techniques in the literature to determine *Eq. (1)* [18]-[24].

The present study extends the Generalized Gauss-Seidel (GGS) method [14] and gives Modified GGS (MGGS) techniques, which can be improved by adding three extra parameters. In addition, we examine the convergence of the novel techniques under appropriate situations.

An overview of this paper is provided below. In Section 2, we discuss the auxiliary results. Section 3 examines the proposed techniques and their convergence. In Sections 4 and Sections 5, we present numerical results as well as conclusions.

2 | Preliminaries

This section provides a helpful lemma and a few notations for later analysis. Throughout this paper, $|A| = (|a_{ij}|)$ and $\|A\|_\infty$ represent the absolute value as well as the infinity norm of $A = (a_{ij}) \in R^{n\times n}$, respectively. In addition, the symbol *'I'* signifies the identity matrix.

Lemma 1 ([31]). Let $x, z \in R^n$. Then $\||z| - |x|\|_\infty \leq \|z - x\|_\infty$.

3 | Proposed Techniques

Here, we briefly describe the MGGS techniques. We have divided this section into two sections. The first section consists of the MGGS I and its convergence, and the second section concerns the MGGS II and its convergence to solve the AVEs.

3.1 | MGGS I for AVE

Let we split the A matrix as

$$A = \frac{1}{\beta} M - N). \tag{2}$$

With

$$M = \Omega + A_D - \lambda A_L,\ N = \Omega + 1 - \beta)A_D + \beta - \lambda)A_L + \beta A_U.$$

Where β and λ are real parameters with $\beta \neq 0$ and $\Omega = \theta I$ where $0 \leq \theta \leq 1$. Also, A_L, A_D and A_U are strictly lower-triangular, diagonal and strictly upper-triangular parts of A. Based on *Eq. (2)*, we can write *Eq. (1)* as follows:

$$\begin{gathered} \frac{1}{\beta} M - N)x - |x| = b, \\ Mx - \beta|x| = Nx + \beta b. \end{gathered} \tag{3}$$

By using the iterative scheme, we define MGGS I as follows:

$$\begin{gathered} Mx^{k+1} - \beta\left|x^{k+1}\right| = Nx^k + \beta b, \\ x^{k+1} = \beta M^{-1}\left|x^{k+1}\right| + M^{-1}\left(Nx^k + \beta b\right), \qquad k = 0,1,2,\dots). \end{gathered} \tag{4}$$

Remark. If $\beta = \lambda = 1$ and $\Omega = 0$, then the MGGS I tranform to the GGS approach [14].

The subsequent theorem specifies the convergence of MGGS I.

Theorem 1. Consider $A = M - N$, which represents a splitting of A with entries in the diagonal of A greater than one. If

$$\left\|M^{-1}N\right\|_{\infty} < 1 - \beta\left\|M^{-1}\right\|_{\infty}. \tag{5}$$

Then the $\left\{x^k\right\}_{k=0}^{\infty}$ sequence obtained from the MGGS I converges to the unique solution $x^{\star}$ of *Eq. (1)*.

Proof. The proof of $\left\|M^{-1}\right\|_{\infty} < 1$ is easy and is skipped here (see Theorem 3 in [14]). Let us consider $x^{\star}$ and $h^{\star}$ to be two distinct solutions of *Eq. (1)* for the purpose of determining the uniqueness of the solution. By applying *Eq. (4)*, we determine

$$x^{\star} = \beta M^{-1}|x^{\star}| + M^{-1}\ Mx^{\star} + \beta b). \tag{6}$$

$$h^{\star} = \beta M^{-1}|h^{\star}| + M^{-1}\ Mh^{\star} + \beta b). \tag{7}$$

Based on *Eq. (6)* and *Eq. (7)*, we can get

$$x^{\star} - h^{\star} = \beta M^{-1}\ |x^{\star}| - |h^{\star}|) + M^{-1}N\ x^{\star} - h^{\star}).$$

According to Lemma 1 and *Eq. (5)*, we have

$$\begin{aligned}
&\|x^{\star} - h^{\star}\|_{\infty} \leq \beta\left\|M^{-1}\right\|_{\infty}\|x^{\star} - h^{\star}\|_{\infty} + \left\|M^{-1}N\right\|_{\infty}\|x^{\star} - h^{\star}\|_{\infty},\\
&\|x^{\star} - h^{\star}\|_{\infty} < \beta\left\|M^{-1}\right\|_{\infty}\|x^{\star} - h^{\star}\|_{\infty} + (1 - \beta\left\|M^{-1}\right\|_{\infty})\|x^{\star} - h^{\star}\|_{\infty},\\
&\|x^{\star} - h^{\star}\|_{\infty} - \beta\left\|M^{-1}\right\|_{\infty}\|x^{\star} - h^{\star}\|_{\infty} < (1 - \beta\left\|M^{-1}\right\|_{\infty})\|x^{\star} - h^{\star}\|_{\infty},\\
&\left(1 - \beta\left\|M^{-1}\right\|_{\infty}\right)\|x^{\star} - h^{\star}\|_{\infty} < \left(1 - \beta\left\|M^{-1}\right\|_{\infty}\right)\|x^{\star} - h^{\star}\|_{\infty},\\
&\|x^{\star} - h^{\star}\|_{\infty} < \|x^{\star} - h^{\star}\|_{\infty}.
\end{aligned}$$

The result is a contradiction. Consequently, $x^{\star} = h^{\star}$.

To determine convergence, we assume that $x^{\star}$ is a unique solution of *Eq. (1)*. Accordingly, from *Eq. (6)* and

$$x^{k+1} = \beta M^{-1}\left|x^{k+1}\right| + M^{-1}\left(Nx^k + \beta b\right).$$

We deduce

$$x^{k+1} - x^{\star} = \beta M^{-1}\left(\left|x^{k+1}\right| - |x^{\star}|\right) + M^{-1}N\left(x^k - x^{\star}\right).$$

Taking the infinity norm, we get;

$$\begin{aligned}
&\left\|x^{k+1} - x^{\star}\right\|_{\infty} \leq \beta\left\|M^{-1}\right\|_{\infty}\left\|x^{k+1} - x^{\star}\right\|_{\infty} + \left\|M^{-1}N\right\|_{\infty}\left\|x^k - x^{\star}\right\|_{\infty},\\
&\left\|x^{k+1} - x^{\star}\right\|_{\infty} - \beta\left\|M^{-1}\right\|_{\infty}\left\|x^{k+1} - x^{\star}\right\|_{\infty} \leq \left\|M^{-1}N\right\|_{\infty}\left\|x^k - x^{\star}\right\|_{\infty},\\
&\left(1 - \beta\left\|M^{-1}\right\|_{\infty}\right)\left\|x^{k+1} - x^{\star}\right\|_{\infty} \leq \left\|M^{-1}N\right\|_{\infty}\left\|x^k - x^{\star}\right\|_{\infty},\\
&\left\|x^{k+1} - x^{\star}\right\|_{\infty} \leq \frac{\left\|M^{-1}N\right\|_{\infty}}{1 - \beta\|M^{-1}\|_{\infty}}\left\|x^k - x^{\star}\right\|_{\infty}.
\end{aligned}$$

This inequality implies that convergence of the approach can be achieved if *Eq. (5)* is fulfilled.

3.2 | MGGS II for AVE

This unit offers the MGGS II. Using *Eq. (1)* and *Eq. (2)*, the MGGS II is expressed as follows:

Ali et al.| Com. Alg. Num. Dim. 1(4) (2022) 130-136

$$Mx^{k+1} - \beta|x^{k+1}| = Nx^{k+1} + \beta b,$$

$$x^{k+1} = \beta M^{-1}|x^{k+1}| + M^{-1}(Nx^{k+1} + \beta b), \qquad k = 0,1,2, \dots).$$

To demonstrate the convergenceiof MGGS II, we apply the following theorem.

Theorem 2. Consider $A = M - N$, which represents a splitting of A with entries in the diagonal terms of A greater than 1. Then the $\{x^k\}_{k=0}^{\infty}$ sequence obtained from the MGGS II converges to the unique solution $x^\star$ of *Eq. (1)*.

Proof. From Theorem 1, the uniqueness follows directly. In order to verify convergence, we can consider the following:

$$\begin{aligned} &x^{k+1} - x^\star = \beta M^{-1}|x^{k+1}| + M^{-1}(Nx^{k+1} + \beta b) - (\beta M^{-1}|x^\star| + M^{-1} Nx^\star + \beta b), \\ &M(x^{k+1} - x^\star) = \beta(|x^{k+1}| - |x^\star|) + N(x^{k+1} - x^\star), \\ &Mx^{k+1} - Nx^{k+1} - \beta|x^{k+1}| = Mx^\star - Nx^\star - \beta|x^\star|, \\ &(M - N)x^{k+1} - \beta|x^{k+1}| = (M - N)x^\star - \beta|x^\star|. \end{aligned} \tag{8}$$

Since

$$A = \frac{1}{\beta} M - N) \Rightarrow \beta A = M - N.$$

As a result, *Eq. (8)* can be expressed as

$$\beta Ax^{k+1} - \beta|x^{k+1}| = \beta Ax^\star - \beta|x^\star|,$$

$$Ax^{k+1} - |x^{k+1}| = Ax^\star - |x^\star|,$$

$$Ax^{k+1} - |x^{k+1}| = b.$$

Thus, x^{k+1} solves the *AVE (1)* system.

4 | Numericaliexperiments

This part gives four examples in order to provide an overview of the performance of the newly formulated techniques from three points of view:

I. The iteration steps (demonstrated as 'Itr').
II. The CPU time in seconds (denoted as 'Time').
III. The residual error (represented as 'RSV').

The term 'RSV' is described as follows:

$$\mathrm{RSV} := \frac{\|Ax^i - |x^i| - b\|_2}{\|b\|_2} \le 10^{-6}.$$

Example 1. Let

$$A = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{pmatrix} \in R^{n\times n}.$$

Compute $b = Ax^\star - |x^\star| \in R^n$, where $x_k^\star = (-1)^k$. Here the starting iterate is zero vector, and we compare both the proposed techniques with the SOR-like approach outlined in [12] (shown as SL) and the special shift splitting approach described in [15] (shown as SS). *Table 1* describes the results.

Table 1. A comparison of example 1 using $\theta = 0.1$, $\beta = 0.98$ and $\lambda = 0.8$.

Methods	n	1000	2000	3000	4000
SL	Itr	18	18	18	18
	Time	3.0156	13.1249	33.9104	65.1345
	RSV	6.12e – 07	6.13e – 07	6.12e – 07	6.14e – 07
SS	Iter	14	14	14	14
	Time	2.8128	9.0954	17.3028	29.1644
	RSV	8.91e – 07	8.92e – 07	8.93e – 07	8.93e – 07
MGGS I	Itr	9	9	9	9
	Time	2.0823	5.3422	12.3724	21.2038
	RSV	9.86e – 07	6.97e – 07	5.69e – 07	4.92e – 07
MGGS II	Itr	7	7	7	7
	Tim	0.9208	1.8422	2.3677	6.2190
	RS	1.82e – 07	1.83e – 07	1.83e – 07	1.83e – 07

Example 2. Let $A = I + M \in R^{n\times n}$ and $b = Ax^\star - |x^\star| \in R^n$ with

$$M = \begin{pmatrix} S & -0.5I & & & \\ -1.5I & S & -0.5I & & \\ & \ddots & S & \ddots & \\ & & \ddots & \ddots & \\ & & \ddots & & -0.5I \\ & & & -1.5I & S \end{pmatrix} \in R^{n\times n}, x_k^\star = \left(-1)^k, k = 1,2,\dots,n\right)^T \in R^n.$$

Where $S = tridiag\ {-1.5}, 4, -0.5) \in R^{q\times q}, I \in R^{q\times q}$ and $n = q^2$. In example 2, the same starting guess and stopping criteria as in [17] are used. We compare the suggested techniques with the AOR procedure [25] and the MTS scheme [17]. We illustrate our results in *Table 2*:

Table 2. A comparison of example 2 using $\theta = 0.2, = \lambda = 0.9$.

Methods	n	100	400	900	1600
AOR	Itr	97	190	336	706
	Time	0.4721	2.8203	3.2174	6.3887
	RSV	9.80e – 07	9.61e – 07	9.73e – 07	9.84e – 07
MTS	Itr	88	157	250	386
	Time	0.4041	1.7953	3.0219	5.7626
	RSV	8.91e – 07	9.65e – 07	9.18e – 07	9.56e – 07
MGGS I	Itr	46	70	91	110
	Time	0.2108	0.3844	0.9534	1.8747
	RSV	8.83e – 07	8.94e – 07	8.17e – 07	8.57e – 07
MGGS II	Itr	23	34	45	55
	Time	0.1722	0.2344	0.7603	1.4200
	RSV	5.08e – 07	9.96e – 07	7.81e – 07	7.51e – 07

Example 3. Let

$$M = \begin{pmatrix} 8 & -1 & & & \\ -1 & 8 & -1 & & \\ & \ddots & 8 & \ddots & \\ & & \ddots & \ddots & \\ & & \ddots & & -1 \\ & & & -1 & 8 \end{pmatrix} \in R^{n\times n}.$$

And $b = Ax^\star - |x^\star|, \in R^n$, where $x_k^\star = (-1)^k$. Based on the same initial guess and ending criteria as explained in [26]. We compare our newly developed techniques with those presented in [26] (signified as SI with ω = 1.0455) as well as the SOR-like strategy offered in [12] (considered as SOR).

Table 3. A comparison of example 3 using $\theta = 0.2$, $\beta = 0.95$ and $\lambda = 0.85$.

Methods	n	1000	2000	3000	4000	5000
SI	Itr	13	13	14	14	14
	Time	3.9928	8.8680	24.4031	51.3946	73.3394
	RSV	6.04e − 07	8.54e − 07	2.33e − 07	2.69e − 07	3.01e − 07
SOR	Itr	12	13	13	13	13
	Time	1.5136	3.3817	6.1262	7.1715	9.5262
	RSV	9.45e − 08	2.69e − 08	3.29e − 08	3.80e − 08	4.25e − 07
MGGS I	Itr	11	11	11	11	11
	Time	1.2718	2.3422	3.0127	4.4824	6.2511
	RSV	9.45e − 08	2.69e − 08	3.80e − 08	3.80e − 07	3.80e − 07
MGGS II	Itr	5	6	6	6	6
	Time	0.45532	0.5203	0.9376	1.2788	3.7826
	RSV	9.29e − 07	1.22e − 08	1.47e − 08	1.68e − 08	3.81e − 07

Based on the results of *Table 1* to *Table 3*, all tested approaches can compute the solution to *Eq. (1)*. Nevertheless, we notice that both the 'Itr' as well as the 'time' of the suggested techniques are better than the other known procedures. To conclude, we can state that our presented techniques are practicable and effective for AVEs.

5 | Conclusion

We have shown two MGGS techniques for determining *AVE (1)* and verified that the offered strategies converge to the *AVE (1)* under proper selections of the parameters. Theoretical estimation and the numerical study of the proposed algorithms indicate that the proposed techniques are applicable for determining AVEs.

Conflict of Interset

The authors have no conflict of interest for this submission.

References

[1] Mangasarian, O. L. (2014). Linear complementarity as absolute value equation solution. *Optimization letters*, *8*(4), 1529-1534. https://doi.org/10.1007/s11590-013-0656-z

[2] Prokopyev, O. (2009). On equivalent reformulations for absolute value equations. *Computational optimization and applications*, *44*(3), 363-372. https://doi.org/10.1007/s10589-007-9158-1

[3] Mezzadri, F. (2020). On the solution of general absolute value equations. *Applied mathematics letters*, *107*, 106462. https://doi.org/10.1016/j.aml.2020.106462

[4] Rohn*, J. (2004). A theorem of the alternatives for the equation Ax+ B|x|= b. *Linear and multilinear algebra*, *52*(6), 421-426. https://doi.org/10.1080/0308108042000220686

[5] Mangasarian, O. L., & Meyer, R. R. (2006). Absolute value equations. *Linear algebra and its applications*, *419*(2-3), 359-367. https://doi.org/10.1016/j.laa.2006.05.004

[6] Ali, R., Khan, M. R., Abidi, A., Rasheed, S., & Galal, A. M. (2021). Application of PEST and PEHF in magneto-Williamson nanofluid depending on the suction/injection. *Case studies in thermal engineering*, *27*, 101329. https://doi.org/10.1016/j.csite.2021.101329

[7] Mao, X., Wang, X., Edalatpanah, S. A., & Fallah, M. (2019). The monomial preconditioned SSOR method for linear complementarity problem. *IEEE access*, *7*, 73649-73655. DOI: 10.1109/ACCESS.2019.2920485

[8] Najafi, H. S., & Edalatpanah, S. A. (2012). A kind of symmetrical iterative methods to solve special class of LCP (M, q). *International journal of applied mathematics and applications*, *4*(2), 183-189.

[9] Ullah, I., Ali, R., Nawab, H., Uddin, I., Muhammad, T., Khan, I., & Nisar, K. S. (2022). Theoretical analysis of activation energy effect on Prandtl–Eyring nanoliquid flow subject to melting condition. *Journal of non-equilibrium thermodynamics*, *47*(1), 1-12. https://doi.org/10.1515/jnet-2020-0092

[10] Najafi, H. S., & Edalatpanah, S. A. (2013). SOR-like methods for non-Hermitian positive definite linear complementarity problems. *Advanced modeling and optimization*, *15*(3), 697-704. https://camo.ici.ro/journal/vol15/v15c10.pdf

[11] Salkuyeh, D. K. (2014). The Picard–HSS iteration method for absolute value equations. *Optimization letters*, *8*(8), 2191-2202. https://doi.org/10.1007/s11590-014-0727-9

[12] Ke, Y. F., & Ma, C. F. (2017). SOR-like iteration method for solving absolute value equations. *Applied mathematics and computation*, *311*, 195-202. https://doi.org/10.1016/j.amc.2017.05.035

[13] Chen, C., Yu, D., & Han, D. (2020). *Optimal parameter for the SOR-like iteration method for solving the system of absolute value equations*. Retrieved from https://doi.org/10.48550/arXiv.2001.05781

[14] Edalatpour, V., Hezari, D., & Salkuyeh, D. K. (2017). A generalization of the Gauss–Seidel iteration method for solving absolute value equations. *Applied mathematics and computation*, *293*, 156-167. https://doi.org/10.1016/j.amc.2016.08.020

[15] Wu, S., & Li, C. (2020). A special shift splitting iteration method for absolute value equation. *AIMS mathematics*, *5*(5), 5171-5183. https://www.aimspress.com/fileOther/PDF/Math/math-05-05-332.pdf

[16] Moosaei, H., Ketabchi, S., & Jafari, H. (2015). Minimum norm solution of the absolute value equations via simulated annealing algorithm. *Afrika matematika*, *26*(7), 1221-1228. https://doi.org/10.1007/s13370-014-0281-8

[17] Fakharzadeh, A., & Shams, N. N. (2020). An efficient algorithm for solving absolute value equations. *Journal of mathematical extension*, *15*. https://ijmex.com/index.php/ijmex/article/view/1393

[18] Ali, R., Kejia, P., & Ali, A. (2020). Two generalized successive overrelaxation methods for solving absolute value equations. *Mathematics, 40*(4), 44-55.

[19] Ali, R., Khan, I., Ali, A., & Mohamed, A. (2022). Two new generalized iteration methods for solving absolute value equations using M-matrix. *AIMS mathematics*, *7*(5), 8176-8187. http://www.aimspress.com/aimspress-data/math/2022/5/PDF/math-07-05-455.pdf

[20] Ali, R. (2022). The solution of absolute value equations using two new generalized gauss-seidel iteration methods. *Computational and mathematical methods*, *2022*. https://doi.org/10.1155/2022/4266576

[21] Ali, R., Ali, A., Alam, M. M., & Mohamed, A. (2022). Numerical solution of the absolute value equations using two matrix splitting fixed point iteration methods. *Journal of function spaces*, *2022*. https://doi.org/10.1155/2022/7934796

[22] Ali, R., Ali, A., & Iqbal, S. (2022). Iterative methods for solving absolute value equations. *The journal of mathematics and computer science*, *26*, 322-329.

[23] Ali, R. (2022). Numerical solution of the absolute value equation using modified iteration methods. *Computational and mathematical methods*, *2022*. https://downloads.hindawi.com/journals/cmm/2022/2828457.pdf

[24] Khan, A., Iqbal, J., Akgül, A., Ali, R., Du, Y., Hussain, A., ... & Vijayakumar, V. (2022). A newton-type technique for solving absolute value equations. *Alexandria engineering journal*. https://doi.org/10.1016/j.aej.2022.08.052

[25] Li, C. X. (2017). A preconditioned AOR iterative method for the absolute value equations. *International journal of computational methods*, *14*(02), 1750016. https://doi.org/10.1142/S0219876217500165

[26] Guo, P., Wu, S. L., & Li, C. X. (2019). On the SOR-like iteration method for solving absolute value equations. *Applied mathematics letters*, *97*, 107-113. https://doi.org/10.1016/j.aml.2019.03.033